

Properties of the solitonic potentials of the heat operator

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Abstract

Properties of the pure solitonic τ -function and potential of the heat equation are studied in detail. We describe the asymptotic behavior of the potential and identify the ray structure of this asymptotic behavior on the x -plane in dependence on the parameters of the potential.

1 Introduction

The Kadomtsev–Petviashvili (KP) equation was derived as a model for small-amplitude, long-wavelength, weakly two-dimensional waves in a weakly dispersive medium [1]. This equation is a (2+1)-dimensional generalization of the celebrated Korteweg–de Vries (KdV) equation, and from the beginning of the 1970s [2, 3] it is known to be integrable. There are two inequivalent versions of the KP equation: KPI and KP II. Here we consider KP II equation

$$(u_t - 6uu_{x_1} + u_{x_1x_1x_1})_{x_1} = -3u_{x_2x_2} \quad (1.1) \quad \boxed{\text{KPII}}$$

(the KPI equation has an opposite sign in the r.h.s.), where $u = u(x, t)$, $x = (x_1, x_2)$ and subscripts x_1 , x_2 and t denote partial derivatives. The KP II equation is integrable since it can be expressed as compatibility condition $[\mathcal{L}, \mathcal{T}] = 0$ of the Lax pair \mathcal{L} and \mathcal{T} , where operator

$$\mathcal{L}(x, \partial_x) = -\partial_{x_2} + \partial_{x_1}^2 - u(x) \quad (1.2) \quad \boxed{\text{heatop}}$$

defines the well known equation of heat conduction, or heat equation for short, and

$$\mathcal{T}(x, \partial_x, \partial_t) = \partial_t + 4\partial_{x_1}^3 - 6u\partial_{x_1} - 3u_{x_1} - 3\partial_{x_1}^{-1}u_{x_2}. \quad (1.3) \quad \boxed{\text{Lax}}$$

The spectral theory of the operator (1.2) was developed in [4–7] in the case of a real potential $u(x)$ rapidly decaying at spatial infinity, which, however, is not the most interesting case, since the KP II equation was just proposed in [1] in order to deal with two dimensional weak transverse perturbation of the one soliton solution of the KdV. In fact, if $u_1(t, x_1)$ obeys KdV, then $u(t, x_1, x_2) = u_1(t, x_1 + \mu x_2 - 3\mu^2 t)$ solves KP II for an arbitrary constant $\mu \in \mathbb{R}$. In particular, KP II admits a one soliton solution of the form

$$u(x, t) = -\frac{(\kappa_1 - \kappa_2)^2}{2} \text{sech}^2 \left[\frac{\kappa_1 - \kappa_2}{2} x_1 + \frac{\kappa_1^2 - \kappa_2^2}{2} x_2 - 2(\kappa_1^3 - \kappa_2^3) t \right], \quad (1.4) \quad \boxed{1\text{-sol}}$$

where κ_1 and κ_2 are real, arbitrary constants.

A spectral theory of the heat operator (1.2) that also includes solitons has to be built. In the case of a potential $u(x)$ rapidly decaying at spatial infinity, according to [4–7], the main tools in building the spectral theory of the operator (1.2), as in the one dimensional case, are the integral equations defining the Jost solutions. However, if the potential $u(x)$ does not decay at spatial infinity, as it is the case when line soliton solutions are considered (see, e.g., (1.4)), these integral equations are ill-defined and one needs a more general approach. In solving the analogous problem for the nonstationary Schrödinger operator, associated to the KPI equation, the extended resolvent approach was introduced [8]. Accordingly, a spectral theory of the KP II equation that also includes solitons has to be investigated using the resolvent approach. In this framework it was possible to develop the inverse scattering transform for a solution describing one soliton on a generic, rapidly decaying background [9], and to study the existence of the (extended) resolvent for (some) multisoliton solutions [10].

However, the general case of N solitons is still open. In particular this is motivated by a complicated asymptotic behavior of the pure solitonic potential on the x -plane. Here we give a detailed description of this behavior in terms of the soliton parameters. The general form of the multisoliton potential was derived in [11] by means of the rational transformations of the scattering data and in [10] by means of the twisting transformation. In [12] we proved that results of both these constructions coincide one with another and also with the form of the multisoliton potential given in terms of the τ -function by Biondini, Kodama et al, see review article [13] and references therein.

The paper is organized as follows. In Sec. 2 we write down the multisoliton potential. They are labeled by the two numbers (topological charges), N_a and N_b , that obey condition

$$N_a, N_b \geq 1, \quad (1.5) \quad \boxed{\text{nanb}}$$

and showing at large spaces N_b “ingoing” rays and N_a “outgoing” rays. By using notations of [12], we give for these potentials two dual representations in terms of τ -functions, which depend on

$$\mathcal{N} = N_a + N_b, \quad (1.6) \quad \boxed{\text{Nnanb}}$$

(so that $\mathcal{N} \geq 2$) real parameters κ_n and on a $N_a \times N_b$ constant matrix d . In Sec. 3 we study the asymptotic behavior at large x of the multisoliton potential in details and show that the round angle at the origin can be divided in \mathcal{N} angular sectors such that along the directions of their bordering rays the potential has constant soliton like behavior, while along directions inside the sectors the potential has an exponential decaying behavior, which we derive explicitly. In particular, for a special subclass of the solitonic potentials we present explicitly the ray structure in terms of the parameters κ_n .

2 Multisoliton potentials

Let we have \mathcal{N} real parameters

$$\kappa_1 < \kappa_2 < \dots < \kappa_{\mathcal{N}}, \quad (2.1) \quad \boxed{\text{kappas}}$$

and introduce the functions

$$K_n(x) = \kappa_n x_1 + \kappa_n^2 x_2, \quad n = 1, \dots, \mathcal{N}. \quad (2.2) \quad \boxed{\text{Kn}}$$

Let

$$e^{K(x)} = \text{diag}\{e^{K_n(x)}\}_{n=1}^{\mathcal{N}}, \quad (2.3) \quad \boxed{\text{eK}}$$

be a diagonal $\mathcal{N} \times \mathcal{N}$ -matrix, \mathcal{D} a $\mathcal{N} \times N_b$ constant real matrix with at least two nonzero maximal minors, and let \mathcal{V} be an “incomplete Vandermonde matrices,” i.e., the $N_b \times \mathcal{N}$ -matrix

$$\mathcal{V} = \begin{pmatrix} 1 & \dots & 1 \\ \vdots & & \vdots \\ \kappa_1^{N_b-1} & \dots & \kappa_{\mathcal{N}}^{N_b-1} \end{pmatrix}. \quad (2.4) \quad \boxed{\text{W}}$$

Then, the soliton potential is given by

$$u(x) = -2\partial_{x_1}^2 \log \tau(x), \quad (2.5) \quad \boxed{\text{ux}}$$

where the τ -function can be expressed as

$$\tau(x) = \det(\mathcal{V} e^{K(x)} \mathcal{D}), \quad (2.6) \quad \boxed{\text{tau}}$$

see the review paper [13], references therein, and [12] where the same notations have been used.

There exists (see [14], [10], and [12]) a dual representation for the potential in terms of the τ -function

$$\tau'(x) = \det(\mathcal{D}' e^{-K(x)} \gamma \mathcal{V}'), \quad (2.7) \quad \boxed{\text{tau}'}$$

where \mathcal{D}' is a constant real $N_a \times \mathcal{N}$ -matrix that like the matrix \mathcal{D} has at least two nonzero maximal minors and that is orthogonal to the matrix \mathcal{D} in the sense that

$$\mathcal{D}' \mathcal{D} = 0, \quad (2.8) \quad \boxed{\text{d12}}$$

being the zero in the r.h.s. a $N_a \times N_b$ -matrix, and where \mathcal{V}' is the $\mathcal{N} \times N_b$ -matrix

$$\mathcal{V}' = \begin{pmatrix} 1 & \dots & \kappa_1^{N_a-1} \\ \vdots & & \vdots \\ 1 & \dots & \kappa_{\mathcal{N}}^{N_a-1} \end{pmatrix}, \quad (2.9) \quad \boxed{\text{sol}'}$$

and γ the constant, diagonal, real $\mathcal{N} \times \mathcal{N}$ -matrix

$$\gamma = \text{diag}\{\gamma_n\}_{n=1}^{\mathcal{N}}, \quad \gamma_n = \prod_{n'=1, n' \neq n}^{\mathcal{N}} (\kappa_n - \kappa_{n'})^{-1}. \quad (2.10) \quad \boxed{\text{gamma}}$$

This dual representation follows if one notices that

$$\tau(x) = (-1)^{N_a N_b + N_a(N_a-1)/2} \left(\prod_{n=1}^{\mathcal{N}} e^{K_n(x)} \right) V(\kappa_1, \dots, \kappa_{N_a+N_b}) \tau'(x), \quad (2.11) \quad \boxed{\text{tautau}}$$

where V denotes the Vandermonde determinant

$$V(\kappa_1, \dots, \kappa_{\mathcal{N}}) = \prod_{1 \leq m < n \leq \mathcal{N}} (\kappa_n - \kappa_m). \quad (2.12) \quad \boxed{\text{V}}$$

Matrices \mathcal{D} and \mathcal{D}' obey rather interesting properties [12]. Taking into account that the matrix \mathcal{D} has nonzero maximal minors we can always write it in the form

$$\mathcal{D} = \pi \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}, \quad (2.13) \quad \boxed{\text{block1}}$$

where π is a $\mathcal{N} \times \mathcal{N}$ -permutation matrix such that the $N_b \times N_b$ -matrix d_2 is nonsingular. Then, we can rewrite this equality in the form

$$\mathcal{D} = \pi \begin{pmatrix} d \\ E_{N_b} \end{pmatrix} d_2 \quad (2.14) \quad \boxed{\text{block}}$$

where $d = d_1 d_2^{-1}$ is a constant real $N_a \times N_b$ -matrix and E_{N_b} the $N_b \times N_b$ unity matrix.

Now, we notice (see [13]) that the expression for the potential in (2.5), as well as condition (2.8), are left unchanged if in (2.6) and (2.7), respectively, we multiply the $\mathcal{N} \times N_b$ -matrix \mathcal{D} from the right by a nonsingular constant $N_b \times N_b$ -matrix and the $N_a \times \mathcal{N}$ -matrix \mathcal{D}' from the left by a nonsingular constant $N_a \times N_a$ -matrix. We conclude that the matrix d_2 is unessential and that the matrix \mathcal{D} can be chosen to have a simple block structure. It is clear, however, that choice of matrixes π and d is not unique as the matrix \mathcal{D} can have different nonzero maximal minors.

Let us write now $\mathcal{D}' \pi = (d'_1, d'_2)$, where d'_1 is $N_a \times N_a$ -matrix and d'_2 is $N_a \times N_b$ -matrix. Then by (2.8) we get $d'_2 = -d'_1 d$, where d is the same $N_a \times N_b$ matrix as in (2.14). Thus, for \mathcal{D}' we get the block structure

$$\mathcal{D}' = d'_1 (E_{N_a}, -d) \pi^\dagger, \quad (2.15) \quad \boxed{\text{block'}}$$

and thanks to equivalence of τ and τ' we get that $\det d'_1 \neq 0$ and, then, also \mathcal{D}' can be chosen to have a simple block structure.

Moreover, by means of the block structures given in (2.14) and (2.15) we get

$$\mathcal{D}^\dagger \mathcal{D} = d_2^\dagger (E_{N_b} + d^\dagger d) d_2, \quad \mathcal{D}' \mathcal{D}'^\dagger = d'_1 (E_{N_a} + d d^\dagger) d'_1{}^\dagger, \quad (2.16) \quad \boxed{\text{d14}}$$

where \dagger denotes the Hermitian conjugation of a matrix (in fact, transposition here). So both matrices $\mathcal{D}^\dagger \mathcal{D}$ and $\mathcal{D}' \mathcal{D}'^\dagger$ are positive and then \mathcal{D} and \mathcal{D}' admit, respectively, left and right inverse (see [15]). Precisely, we have

$$(\mathcal{D})^{(-1)} \mathcal{D} = E_{N_b}, \quad \mathcal{D}' (\mathcal{D}')^{(-1)} = E_{N_a}, \quad (2.17) \quad \boxed{\text{d17}}$$

with

$$(\mathcal{D})^{(-1)} = (\mathcal{D}^\dagger \mathcal{D})^{-1} \mathcal{D}^\dagger, \quad (\mathcal{D}')^{(-1)} = \mathcal{D}'^\dagger (\mathcal{D}' \mathcal{D}'^\dagger)^{-1}. \quad (2.18) \quad \boxed{\text{d15}}$$

Products of these matrices in the opposite order give the real Hermitian $\mathcal{N} \times \mathcal{N}$ -matrixes

$$P = \mathcal{D} (\mathcal{D})^{(-1)} = \mathcal{D} (\mathcal{D}^\dagger \mathcal{D})^{-1} (\mathcal{D})^\dagger, \quad (2.19) \quad \boxed{\text{d19}}$$

$$P' = (\mathcal{D}')^{(-1)} \mathcal{D}' = (\mathcal{D}')^\dagger (\mathcal{D}' \mathcal{D}'^\dagger)^{-1} \mathcal{D}', \quad (2.20) \quad \boxed{\text{d18}}$$

which are orthogonal projectors, i.e.,

$$P^2 = P, \quad (P')^2 = P', \quad P P' = 0 = P' P, \quad (2.21) \quad \boxed{\text{d22}}$$

and complementary in the sense that

$$P + P' = E_{\mathcal{N}}. \quad (2.22) \quad \boxed{\text{d23}}$$

Orthogonality of the projectors follows from (2.8) and the last equality from obvious relations of the kind $(E_{N_b} + d^\dagger d)^{-1} d^\dagger = d^\dagger (E_{N_a} + d d^\dagger)^{-1}$.

Finally, let us mention that if all maximal minors of the matrix \mathcal{D} are nonzero, the permutation π in (2.14) and (2.15) can be arbitrarily chosen. Then, we can chose it equal to the identity, getting the simplified representations

$$\mathcal{D} = \begin{pmatrix} d \\ E_{N_b} \end{pmatrix} d_2, \quad \mathcal{D}' = d'_1(E_{N_a}, -d), \quad (2.23) \quad \boxed{\text{blockreg}}$$

so that in this special case without loss of generality we choose

$$\mathcal{D} = \begin{pmatrix} d \\ E_{N_b} \end{pmatrix}, \quad \mathcal{D}' = (E_{N_a}, -d). \quad (2.24) \quad \boxed{\text{block2}}$$

In order to study the properties of the potential, it is convenient to use an explicit representation for the determinant (2.6). By using the Binet–Cauchy formula for the determinant of a product of matrices we get

$$\tau(x) = \sum_{1 \leq n_1 < n_2 < \dots < n_{N_b} \leq \mathcal{N}} f_{n_1, \dots, n_{N_b}} \prod_{l=1}^{N_b} e^{K_{n_l}(x)}, \quad (2.25) \quad \boxed{\text{tauf}}$$

with coefficients $f_{n_1, n_2, \dots, n_{N_b}}$ given by

$$f_{n_1, n_2, \dots, n_{N_b}} = V(\kappa_{n_1}, \dots, \kappa_{n_{N_b}}) \mathcal{D}(n_1, \dots, n_{N_b}). \quad (2.26) \quad \boxed{\text{f}}$$

Here we used notation (2.12) for the Vandermonde determinant and notation

$$\mathcal{D}(n_1, \dots, n_{N_b}) = \det \begin{pmatrix} \mathcal{D}_{n_1, 1} & \dots & \mathcal{D}_{n_1, N_b} \\ \vdots & & \vdots \\ \mathcal{D}_{n_{N_b}, 1} & \dots & \mathcal{D}_{n_{N_b}, N_b} \end{pmatrix}, \quad (2.27) \quad \boxed{\text{Do}}$$

for the maximal minors of the matrix \mathcal{D} . Notice that these coefficients are invariant under permutations of the indices.

From (2.25) it follows directly that condition

$$f_{n_1, \dots, n_{N_b}} \geq 0, \quad \text{for all } 1 \leq n_1 < n_2 < \dots < n_{N_b} \leq \mathcal{N}, \quad (2.28) \quad \boxed{\text{reg}}$$

is sufficient (see [13]) for the regularity of the potential $u(x)$, i.e., for the absence of zeros of $\tau(x)$ on the x -plane. Thanks to (2.1), (2.12) and (2.26) this condition is equivalent to the condition that all maximal minors of the matrix \mathcal{D} are non negative. In [12] it was mentioned that condition (2.28) is also necessary for the regularity of a potential under evolution with respect to an arbitrary number of higher times of the KP hierarchy. In [13] it is suggested to decompose soliton solutions of KP II into subclasses, associated to the Schubert cell on Grassmanian. It is proved there that condition (2.28) is necessary for the regularity of the all solutions associated to a cell. However, problem of finding of the conditions necessary for the regularity of the many soliton solution under evolution with respect to KP II only is still opened.

If the matrix \mathcal{D} can be chosen as in (2.24), then introducing new $N_a \times N_b$ -matrix \tilde{d} as

$$\tilde{d}_{n,l} = d_{N_a+1-n,l}(-1)^{l+1}, \quad n = 1, \dots, N_a, \quad l = 1, \dots, N_b, \quad (2.29) \quad \boxed{\text{dd2}}$$

we easily get by (2.27) that

$$\mathcal{D}(n_1, \dots, n_{N_b}) = \det \|\tilde{d}_{n_i, l}\|,$$

$$\text{where } i = 1, \dots, k, \quad l = 1, \dots, N_b, \quad l \neq n_j - N_a, \quad j = k+1, \dots, N_b, \quad (2.30) \quad \boxed{\text{dd1}}$$

and where number k , $1 \leq k \leq N_b$ is defined by a condition $n_k \leq N_a < n_{k+1}$. This proves that if we require condition (2.28) on coefficients $f_{n_1, \dots, n_{N_b}}$, then the matrix \tilde{d} is a totally nonnegative (positive, if all inequalities in (2.28) are strict) one (see ([16, 17])).

In what follows it is convenient to continue indexes of κ_n , $f_{n_1, \dots, n_{N_b}}$, etc, periodically by \mathcal{N} on the whole \mathbb{Z} by means of condition:

$$n \rightarrow n \pmod{\mathcal{N}}. \quad (2.31) \quad \boxed{\text{modN}}$$

Then Eq. (2.25) can be written as

$$\tau(x) = \sum_{n \leq n_1 < n_2 < \dots < n_{N_b} \leq \mathcal{N} + n - 1} f_{n_1, \dots, n_{N_b}} \prod_{l=1}^{N_b} e^{K_{n_l}(x)}, \quad (2.32) \quad \boxed{\text{taufn}}$$

where the r.h.s. is independent of n , and where coefficients $f_{n_1, \dots, n_{N_b}}$ are defined by (2.26) taking (2.31) into account. Thanks to (2.1) and (2.31) we can prove the following lemma.

lemma1 **Lemma 2.1** *Let*

$$g_{l,m,n} = (\kappa_l - \kappa_m)(\kappa_n - \kappa_{n+N_b})(\kappa_l + \kappa_m - \kappa_n - \kappa_{n+N_b}). \quad (2.33) \quad \boxed{111}$$

Then

$$g_{l,m,n} \geq 0 \quad \text{for any } n \in \mathbb{Z}, \quad l = n, \dots, n + N_b, \quad m = n + N_b, \dots, \mathcal{N} + n, \quad (2.34) \quad \boxed{112}$$

and equality takes place only when l and m independently take values n or $n + N_b$ by $\text{mod } \mathcal{N}$.

Proof. Thanks to (2.31) it is enough to prove the lemma for $n = 1, \dots, \mathcal{N}$. For these values of n the κ_n 's are ordered as in (2.1). For other values the ordering is obtained by coming back by means of (2.31) to values belonging to the interval $n = 1, \dots, \mathcal{N}$. Thus, when $n = 1, \dots, N_a$ then n , $n + N_b$ and l are not more than \mathcal{N} (see (1.6)), but instead of the interval for m given in (2.34) we get two intervals: $m = n + N_b, \dots, \mathcal{N}$ and $m = 1, \dots, n$. Then for given values of n , l and m in the first interval lemma follows because the first factor in (2.33) is nonpositive, the second one negative and the third one nonnegative. Zero of the first factor is possible only for $l = m$ and then $l = m = n + N_b$ and of the third factor only when both $l = n$ and $m = n + N_b$ (all equalities by $\text{mod } \mathcal{N}$). In the second case the first factor is nonnegative (zero only at $l = m = n \pmod{\mathcal{N}}$), the second one is negative and the third one is nonpositive (zero only at $l = n + N_b$, $m = n$). In the interval $n = N_a + 1, \dots, \mathcal{N}$ we write by (2.31) $\kappa_{n+N_b} = \kappa_{n-N_a}$, so that again n and $n - N_a$ belong to the interval $1, \dots, \mathcal{N}$. Now the interval of l given in (2.34) decouples in two intervals n, \dots, \mathcal{N} and $1, \dots, n - N_a$, while $m = n - N_a, \dots, n$, so it is positive and less than \mathcal{N} . Then for these values of n , m and l in the first interval thanks to (2.1) the first factor in (2.33) is nonnegative (zero only when $l = m = n \pmod{\mathcal{N}}$), the second factor is positive, and the third one is nonnegative (zero only when $l = n$ and $m = n + N_b$). In the case of the second interval the first factor is nonpositive (zero only when $l = m = n - N_a = n + N_b \pmod{\mathcal{N}}$), the second factor is positive and the third factor is nonpositive (zero only when $l = n - N_a = n + N_b \pmod{\mathcal{N}}$ and $m = n$). ■

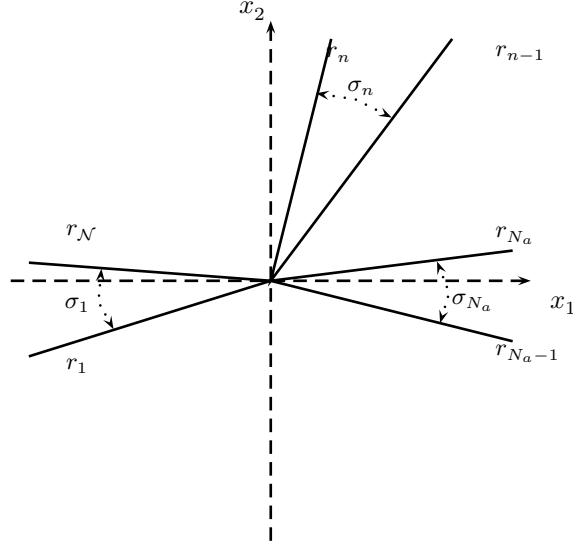


Figure 1: Sector σ_n corresponds to $N_a + 2 \leq n \leq \mathcal{N} - 1$.

fig

3 Asymptotic behavior of the τ -function and potential $u(x)$

Here we study the asymptotic behavior of the function $\tau(x)$ that by (2.25) is determined by interrelations between functions $K_n(x)$ for different n . Since by (2.2)

$$K_l(x) - K_m(x) = (\kappa_l - \kappa_m)(x_1 + (\kappa_l + \kappa_m)x_2), \quad \text{for any } l, m \in \mathbb{Z}, \quad (3.1)$$

linen

that is, these differences are linear with respect to the space variables, the asymptotic behavior must have sectorial structure on the x -plane. In order to describe this structure we introduce rays r_n given by

$$r_n = \{x : x_1 + (\kappa_{n+N_b} + \kappa_n)x_2 = 0, (\kappa_{n+N_b} - \kappa_n)x_2 < 0\}, \quad n = 1, \dots, \mathcal{N}, \quad (3.2)$$

rayn

that intersect the lines $x_2 = \pm 1$, respectively, at the points

$$\begin{aligned} &(\kappa_{n+N_b} + \kappa_n, -1) && \text{for } n = 1, \dots, N_a \\ &(-\kappa_{n+N_b} - \kappa_n, 1) && \text{for } n = N_a + 1, \dots, \mathcal{N}. \end{aligned} \quad (3.3)$$

y0n

Thanks to (2.1) and (2.31),

$$\kappa_{n+N_b} - \kappa_n > 0, \quad n = 1, \dots, N_a, \quad (3.4)$$

ineq1

$$\kappa_{n+N_b} - \kappa_n < 0, \quad n = N_a + 1, \dots, \mathcal{N}, \quad (3.5)$$

ineq2

therefore, increasing n from 1 to N_a the ray rotates anticlockwise in the lower half x -plane, crosses the positive part of the x_1 -axis in coming to $n = N_a + 1$ and, then, rotates anticlockwise in the upper half x -plane up to $n = \mathcal{N}$ and, finally, crossing the negative part of the x_1 -axis, for $n = \mathcal{N} + 1$ comes back to the ray r_1 thanks to (2.31), see Fig. 1.

Let us assume that some rays, say, r_m and r_n , where for definiteness $m < n$, are parallel. By (3.2) this means that $\kappa_{m+N_b} + \kappa_m = \kappa_{n+N_b} + \kappa_n$, i.e., that $\kappa_n - \kappa_m = \kappa_{m+N_b} - \kappa_{n+N_b}$,

where the l.h.s. is positive thanks to (2.1). Then, because of (2.31) it is easy to see that the r.h.s. can be positive only if $1 \leq m \leq N_a < n \leq \mathcal{N}$, so that by (3.2) ray r_m is in the bottom halfplane and r_n in the upper one. Thus rays can be parallel only if they belong to different halfplanes, but in the case $N_a \neq N_b$ this is possible for a special choice of the parameters κ_n only. On the contrary, in the case $N_a = N_b$ all pairs r_n and r_{n+N_a} , $n = 1, \dots, N_a$, and only these pairs give parallel rays. In the special case $N_a = N_b = 1$ we get two rays producing the straight line $x_1 + (\kappa_1 + \kappa_2)x_2 = 0$ that divides the x -plane in two halfplanes.

Now we introduce sectors σ_n , which are subsets of the x -plane characterized as

$$\sigma_n = \{x : K_{n-1}(x) < K_{n+N_b-1}(x) \text{ and } K_n(x) > K_{n+N_b}(x)\}, \quad n = 1, \dots, \mathcal{N}. \quad (3.6)$$

signan

Thanks to (3.1) and the discussion above it follows that the sectors σ_n are sharp (for $\mathcal{N} > 2$) angular sectors with vertexes at the origin of the coordinates bounded from the right (looking from the origin) by the ray r_{n-1} and from the left (looking from the origin) by the ray r_n . With the increasing of n sectors σ_n are ordered anticlockwise, starting “from the left” with the sector σ_1 , that includes the negative part of the x_1 -axis, and, then, with the sectors σ_n ($n = 2, \dots, N_a$) in the bottom half-plane, the sector σ_{N_a+1} “to the right”, that includes the positive part of the x_1 -axis, and the sectors σ_n ($n = N_a + 2, \dots, \mathcal{N}$) on the upper half-plane, finishing with the sector $\sigma_{\mathcal{N}}$ tangent to the sector σ_1 , covering in this way the whole x -plane with the exception of the bordering rays r_n . Therefore, the sectors σ_n define a \mathcal{N} -fold discretization of the round angle at the origin, with n playing the role of a discrete angular variable, see Fig. 1. It is clear that in the study of asymptotic behavior of τ -function we consider x -plane as vector space, as finite part of x when $x \rightarrow \infty$ is irrelevant. For determining the directions of rays r_n and sectors σ_n , we introduce the vectors

$$y_n = (\kappa_{n+N_b}^2 - \kappa_n^2, \kappa_n - \kappa_{n+N_b}), \quad n = 1, \dots, \mathcal{N}, \quad (3.7)$$

pointn'

that enables us to give the following definition.

def1'

Definition 3.1 We say that $x \rightarrow \infty$ along the ray r_n if $x \rightarrow \infty$ and there exists such function $\alpha(x) \rightarrow +\infty$ that $x - \alpha(x)y_n$ is bounded. This will be denoted as $x \xrightarrow{r_n} \infty$.

We say that $x \rightarrow \infty$ in the sector σ_n , $n = 1, \dots, \mathcal{N}$, if $x \rightarrow \infty$ and there exist such functions $\alpha(x) \rightarrow +\infty$ and $\beta(x) \rightarrow +\infty$ that $x - \alpha(x)y_{n-1} - \beta(x)y_n$ is bounded. This will be denoted as $x \xrightarrow{\sigma_n} \infty$.

Notice that for $x \xrightarrow{r_n} \infty$

$$K_n(x) - K_{n+N_b}(x) \text{ is bounded and } (\kappa_{n+N_b} - \kappa_n)x_2 \rightarrow -\infty, \quad (3.8)$$

def12

and for $x \xrightarrow{\sigma_n} \infty$

$$K_{n+N_b-1}(x) - K_{n-1}(x) \rightarrow +\infty, \quad K_n(x) - K_{n+N_b}(x) \rightarrow +\infty, \quad (3.9)$$

def11

as follows directly from the definition. In fact we can prove a more general statement.

lemma3

Lemma 3.1 Let $N_a, N_b \geq 1$, and $n \in \mathbb{Z}$ arbitrary. Then

1. if $x \xrightarrow{r_n} \infty$ then $K_l(x) - K_m(x) \rightarrow +\infty$ or bounded for any $l = n, \dots, n + N_b$ and $m = n + N_b, \dots, \mathcal{N} + n$, where boundedness takes place if and only if $(l, m) = (n, n), (n, n + N_b), (n + N_b, n), (n + N_b, n + N_b)$;
2. if $x \xrightarrow{\sigma_n} \infty$ then $K_l(x) - K_m(x) \rightarrow +\infty$ for any $l = n, \dots, n + N_b - 1$ and $m = n + N_b, \dots, \mathcal{N} + n - 1$;

where summation in indexes is always understood by $\text{mod } \mathcal{N}$.

Proof. Thanks to (3.1) and (3.7) we have that $K_l(y_n) - K_m(y_n) = g_{l,m,n}$. Then the first statement of the lemma follows directly from Definition 3.1 and Lemma 2.1. In the same way we get that $K_l(x) - K_m(x) = \alpha g_{l,m,n-1} + \beta g_{l,m,n}$ under substitution $x = \alpha y_{n-1} + \beta y_n$. Then the second statement of the lemma follows from the second statement of the Lemma 2.1 while intervals for l and m appear as intersections of the corresponding intervals in (2.34) for n and $n \rightarrow n-1$. ■

Now we can prove the following Theorem.

th1 **Theorem 3.1** *The asymptotic behavior of $\tau(x)$ for $x \rightarrow \infty$ is given by*

$$x \xrightarrow{r_n} \infty : \quad \tau(x) = (z_n + z_{n+1} e^{K_{N_b+n}(x) - K_n(x)} + o(1)) \exp \left(\sum_{j=n}^{n+N_b-1} K_j(x) \right), \quad (3.10) \quad \boxed{3:232}$$

$$x \xrightarrow{\sigma_n} \infty : \quad \tau(x) = (z_n + o(1)) \exp \left(\sum_{l=n}^{n+N_b-1} K_l(x) \right), \quad (3.11) \quad \boxed{\text{tauasympt}}$$

for any $n \in \mathbb{Z}$, where coefficients z_n are defined as

$$z_n = f_{n,n+1,\dots,n+N_b-1} \equiv V(\kappa_n, \dots, \kappa_{n+N_b-1}) \mathcal{D}(n, \dots, n+N_b-1), \quad (3.12) \quad \boxed{\text{zn}}$$

(cf. (2.26)). Terms $o(1)$ are decaying exponentially.

Proof. Using representation (2.32) we can write

$$\begin{aligned} \exp \left(- \sum_{l=n}^{n+N_b-1} K_l(x) \right) \tau(x) &= \\ &= \sum_{n \leq n_1 < n_2 < \dots < n_{N_b} \leq \mathcal{N} + n - 1} f_{n_1, \dots, n_{N_b}} \exp \left(\sum_{j=1}^{N_b} K_{n_j}(x) - \sum_{l=n}^{n+N_b-1} K_l(x) \right). \end{aligned} \quad (3.13)$$

Let us consider first the asymptotics when $x \xrightarrow{\sigma_n} \infty$. Then each term $K_{n_j}(x)$ either cancels with some term in the last sum, when the index n_j equals by $\text{mod } \mathcal{N}$ one index l in the interval $\{n, \dots, n+N_b-1\}$, i.e., an interval of the index l in Lemma 3.1, or belong to the interval $\{n+N_b, \dots, \mathcal{N}+n-1\}$. Thus by Lemma 3.1 all these exponents at infinity are negative with the only exception of the case where $\{n_1, \dots, n_{N_b}\} = \{n, \dots, n+N_b-1\} \pmod{\mathcal{N}}$. This proves (3.11). Proof of (3.10) goes in the same way with the only difference that we use the first statement of the Lemma 3.1. Statement on the asymptotic behavior of the terms of $o(1)$ kind is obvious by construction. ■

As we see from (3.10) and (3.11) the leading asymptotic behavior is fixed by this theorem only if all

$$z_n > 0, \quad n = 1, \dots, \mathcal{N}. \quad (3.14) \quad \boxed{\text{zn0}}$$

This condition is a sufficient for the regularity of the potential at large x , while specific examples show that it is not sufficient for the regularity of the potential in all the x -plane. We also see that the asymptotics along the ray direction r_n is given by the sum of the leading terms obtained for $x \xrightarrow{\sigma_n} \infty$ and $x \xrightarrow{\sigma_{n+1}} \infty$. Since the exponential factor cancels out when expansion (3.10) of $\tau(x)$ is inserted in (2.5), the factor in parenthesis gives the ray behavior of the potential $u(x)$ at infinity. Thus, if condition (3.14) is imposed potential $u(x)$ has

exactly \mathcal{N} rays r_n , obeying nontrivial asymptotic behavior of onesoliton potential (cf. (1.4)) with parameters κ_n and κ_{n+N_b} . Explicitly, taking into account (2.31) we get the following ray structure and asymptotic behavior of the potential along rays:

$$\begin{aligned} x_2 \rightarrow -\infty, \quad x_1 + (\kappa_n + \kappa_{n+N_b})x_2 \text{ being bounded:} \\ u(x) \cong -2\partial_{x_1}^2 \log(z_n + z_{n+1}e^{K_{n+N_b}(x)-K_n(x)}), \quad n = 1, \dots, N_a. \end{aligned} \quad (3.15) \quad \boxed{\text{down}}$$

and

$$\begin{aligned} x_2 \rightarrow +\infty, \quad x_1 + (\kappa_n + \kappa_{n+N_a})x_2 \text{ being bounded:} \\ u(x) \cong -2\partial_{x_1}^2 \log(z_{n+N_a} + z_{n+N_a+1}e^{K_n(x)-K_{n+N_a}(x)}), \quad n = 1, \dots, N_b, \end{aligned} \quad (3.16) \quad \boxed{\text{up}}$$

i.e., N_a asymptotic rays in the bottom half plane and N_b asymptotic rays in the upper half-plane. Asymptotic behavior inside sectors σ_n is given by (3.11) where the only x -dependent term is the exponential factor. Taking into account its linear dependence on x and (3.14) we get that $u(x)$ decays exponentially in all directions inside all sectors.

4 Concluding remarks

Asymptotic behavior of the τ -function and potential derived here is based on condition (3.14) allowing vanishing or negative values of coefficients in (2.25) different from z_n (see (3.12)). Thanks to (3.14) we were able to find explicitly that the ray directions are given by the sums $\kappa_n + \kappa_{n+N_b}$, with $n = 1, \dots, \mathcal{N}$, and to derive the exact asymptotic behavior of the τ -function on the x -plane. Let us mention that analogous result was derived in [18] for a very special subclass of solitonic potentials, given by matrix \mathcal{D} in (2.6) equal to a product of a diagonal matrix by the matrix transpose to \mathcal{V} . As we mentioned above, condition (3.14) does not guarantee (in contrast to (2.28)) absence of singularities of the potential $u(x)$. But (3.10) and (3.11) prove that thanks to (3.14) singularities of the potential (2.5) cannot appear in asymptotics, as inside sectors $u(x) \rightarrow 0$ and along rays r_n it is given by (3.15) and (3.16), that is also finite thanks to (3.14). Let us emphasize that parameters N_a and N_b giving numbers of rays of the τ -function and potential in the bottom and upper half-planes, correspondingly, appeared in [10] as numbers of poles of the Jost and dual Jost solutions. And, finally, we mention that Theorem 3.1 is also valid when all inequalities in the regularity requirement (2.28) are strict, i.e., all maximal minors of the matrix \mathcal{D} are positive, as this condition is stronger than (3.14) that was enough to prove the theorem.

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